EXPLORING MATHEMATICS ON YOUR OWN

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The present article illustrates that how mathematics can be explored by an individual. As mathematics relies on logic and reasoning, the reasoning has been dealt in two ways, namely; inductive and deductive. These two have been shown through activities. The article also discusses about various branches of mathematics through some interesting examples, patterns, conjectures and also shows use of mathematics in every walk of life. It also highlights the historical development of some important disciplines/concepts in mathematics.

This is Mathematics

Why has Mathematics become so important in recent years? Why are industrialists and politicians so concerned about the shortage of Mathematicians? Can the new electronic brains solve our mathematical problems faster and more accurately than a person and eliminate the need for Mathematicians?

To answer these questions, we need to know what Mathematics is and how it is used. Mathematics is much more than arithmetic, which is the science of numbers and calculations. It is more than algebra, which is the language of symbols, operations and relations. It is much more than geometry, which is the study of shapes, sizes and spaces. It is more than statistics, which is the science of interpreting data and graphs. It is more than calculus, which is the study of change, limits and infinity. Mathematics includes all of these and much more.

Mathematics is a way of thinking and reasoning. Mathematics can be used to determine whether or not an idea is true, or at least, whether it is probably true. Mathematics is a field of exploration and invention, where new ideas are being discovered everyday. It is the way of thinking that is used to solve all kinds of problems in the fields of science, government and industries. It is a language of symbols that is understood in all civilised nations of the world. Also, Mathematics is a language of precision and accuracy. It has even been suggested that Mathematics would be one of the language that would be understood by inhabitants of Mars (if there are any). It is an art like music, with symmetry, pattern and rhythm that can be very pleasing.

Mathematics has also been described as the study of patterns, where pattern is any kind of regularity in form or idea. This study of patterns has been very important for science because pattern, regularity and symmetry occur so often in nature. For example, light, sound, electric currents, magnetism, waves of the sea, the flight of a plane, the shape of a snowflake and the mechanics of the atom all have patterns that can be classified by Mathematics.

Mathematics in Our World

If we look back at the story of civilisation, we will find that mathematics has always played a major role.

It has been the means of:

- measuring property boundaries
- predicting the seasons
- navigating ships
- building homes and bridges
- drawing maps
- developing weapons and planning warfare
- understanding the motion of heavenly bodies
- increasing trade and commerce.

During our lifetime, mathematics has been the tool for:

- discovering new scientific principles
- inventing new machines
- creating electronic brains
- developing strategy in games
- directing traffic and communications
- making new vaccines and medicines
- harmless atomic energy
- navigating in space
- discovering new ores
- forecasting the weather
- predicting population growth.

Not everyone can be mathematician or a scientist, but in order to understand our modern world, it is necessary to know something about mathematics. This knowledge should make you more successful in school, at home, or in your future vocation. Just to be a good citizen in a nation when all these changes are taking place will require a knowledge of some mathematics. Certainly, people in our government must be mathematically informed, if they have to make wise decisions in our complex world of new ideas. Of course, if you are interested in a career in science, statistics or engineering, which are based on mathematics, you will need to become an expert in the field. Today, there is a great demand for mathematicians to do research, teach or find new applications of mathematics. The professional mathematicians of the world has often played an important role in the building of our civilisation. The methods of reasoning used by the great mathematicians of the world and the products of their logic are more important than ever in our present culture.

About Arithmetic

We are aware that arithmetic is a logical mathematical system. But arithmetic began as a language to answer questions about man's everyday life. To answer these questions, it was necessary to invent numbers and measures and a way of combining and comparing these numbers. This finally led to the development of the science of numbers of arithmetic.

It took a man a long time to invent a workable system of symbols for numbers. These symbols for numbers are called numerals. People worked for a long time before a zero symbol was invented. We know that our base ten numeration system, which uses the place value principle, is much better than many ancient systems of numeration. But our system is not only the system, nor it is necessarily the best one. Some people feel that a numeration system using twelve basic symbols would be superior to our base ten system. Another numeration system, called the binary system, which uses only two symbols, 0 and 1, has found use in new computers and other electronic devices.

Binary Numbers

0 zero	100 four
1 one	101 five
10 two	110 six
11 three	111 seven

The study of numbers has always been a fascinating topic. Even the most common numerical problem can suggest new questions or ideas. See if you can explain the method used in solving the following division problem:

```
23) 552
    230 = 10 × 23
    ---------------
    322
    230 = 10 × 23
    ---------------
    92
    92 =  4 × 23
    ---------------
          24
```

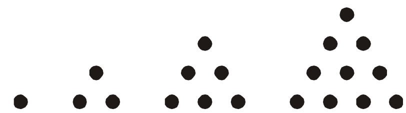
$552 \div 23 = 10 + 10 + 4 = 24$

We can often find unusual numerical patterns that give new insight into number relations. A good illustration is the pattern of multiples of 9.

1×9	= 9	$0 + 9 = 9$
2×9	= 18	$1 + 8 = 9$
3×9	= 27	$2 + 7 = 9$
4×9	= 36	$3 + 6 = 9$
5×9	= 45	$4 + 5 = 9$
6×9	= 54	$5 + 4 = 9$
7×9	= 63	$6 + 3 = 9$
8×9	= 72	$7 + 2 = 9$
9×9	= 81	$8 + 1 = 9$
10×9	= 90	$9 + 0 = 9$

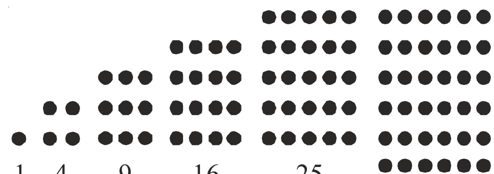
Note how the tens digit goes up from 1 to 9 while the units digit goes down from 9 to 0. After 45, the products are the previous numerals written backwards.

There are many other interesting number patterns. For instance, triangular patterns can be used to represent numbers like 1, 3, 6 and 10.



1 3 6 10

Then there are patterns of squares that represent numbers like 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

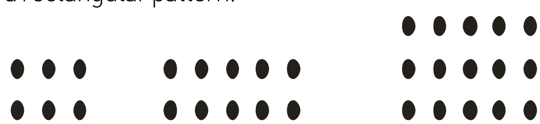


1 4 9 16 25 36

A strange relationship for square numbers is that they are equal to the sum of consecutive odd integers (whole numbers), like this:

$$4 = 1 + 3$$
$$9 = 1 + 3 + 5$$
$$16 = 1 + 3 + 5 + 7$$

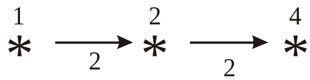
Some numbers, like 6, 10, 15 can represented with a rectangular pattern.



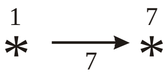
$2 \times 3 = 6$

Some numbers, such as 2, 3, 5, 7, 11 and 13, cannot be used to form either squares or rectangles. These numbers have no divisors except themselves and 1 and are called prime numbers. Mathematicians have been trying for many years to find relationships between prime numbers so that they could write a formula to describe these numbers and tell how to find them. No one has yet found a formula that will give all the prime numbers. Hence a good part of the work with prime numbers is still inductive.

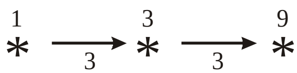
Let us examine some interesting prime number patterns. If we use an arrow (⟶) to represent a prime number as a multiplier, and asterisk (*) to represent a product, we can make drawings to show that every whole number is the product of prime numbers beginning with 1.



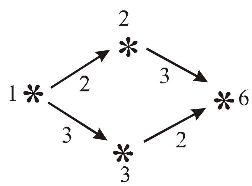
This diagram tells us that 1 × 2 = 2 and 2 × 2 = 4. The pattern for 7 is



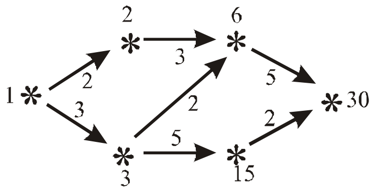
While the pattern for 9 is

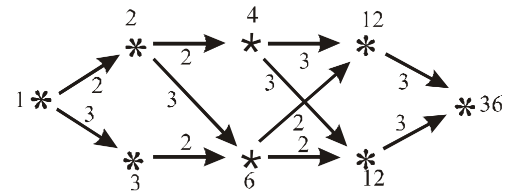


But, 6 can have two different paths, like this:



Note that the length of the arrows is not related to the size of the multiplier. Other patterns might look like this:





What kind of numbers have a straight line as a pattern? What kind of numbers have a complex figure as a pattern?

These examples illustrate how the patterns found in number relationships are of interest to mathematicians. The theory of numbers, a field of mathematics concerned with the study of number patterns, has been responsible for the discovery of many new mathematical ideas. We see that simple arithmetic can be the basis for many intriguing mathematical ideas and relationships.

Here, we would like to give some examples based on reasoning.

Example 1: There are between 50 and 60 eggs in a basket. If I count them out 3 at a time, I have 2 left over, but if I count them out 5 at a time, I have 4 left over. How many eggs are there in the basket?

Answer: 59 eggs

Example 2: Three men who share a food are charged ₹ 10 each or ₹ 30 in all. The manager, after some reflection, decides that he has overcharged them, since they are sharing the food, so he gives a servant ₹ 5 to return to them. The servant, not being able to divide ₹ 5 into three equal parts, pockets ₹ 2 for himself and returns only ₹ 1 to each man. That makes the food cost for each man ₹ 9 or 27 for all three. If we add this ₹ 27 to the ₹ 2 the servant keeps, we get a total of ₹ 29. Yet the men paid ₹ 30. Where is other ₹ 1?

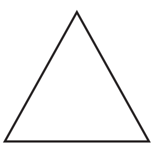
The men paid ₹ 27, of which the manager got ₹ 25 and the servant ₹ 2.

Example 3: A king who wishes to choose a prime minister decides to test the mentality of the three top candidates for the position. He tells the candidates that he will blindfold each one and then mark either a red cross or a blue cross on the forehead of each. He will then remove the blindfolds. Each candidate has to raise his hand, if he sees a red cross and drop his hand when he determines the colour of his own cross. The king, then blindfolds each candidate and proceeds to mark a red cross on each forehead. He then removes the blindfolds. After looking at each other, the prospective prime ministers raise their hands. After a short time, one candidate lowers his hand and says, 'My cross is red', and gives his reasons. Can you duplicate his reasoning?

Suppose the candidates are A, B and C. A reasons that if he had a blue cross, then B or C would immediately know that they had a red cross. Since B and C would not determine their colour, A must also be marked with a red cross. This is an example of indirect reasoning.

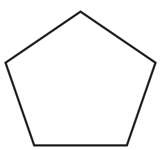
The World of Mathematics

Let us try an experiment with geometrical figures. Draw several geometrical figures with different numbers of sides, as shown in the following figure:

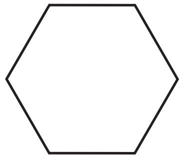


Triangle

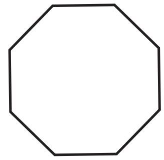
Rectangle



Pentagon



Hexagon



Octagon

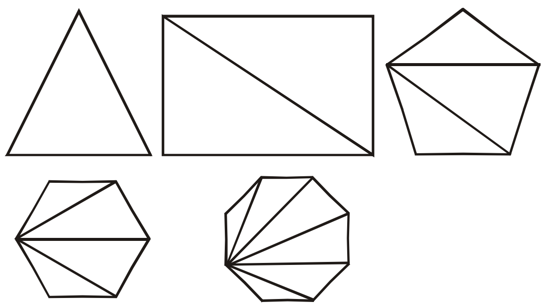
Measure the angles of these figures. We should get results like these:

Figures	No. of sides (n)	Sum of internal angles (S)	No. of st. angles (180°) in each sum
Triangle	3	180°	1
Rectangle	4	360°	2
Pentagon	5	540°	3
Hexagon	6	720°	4
Octagon	8	1080°	6

How does the number of sides of each figure compare with the number of straight angles? This table suggests the following conclusion: 'The sum of the measures of the angles of a geometrical figure equals the product of 180° multiplied by 2 less than the number of sides ($n-2$)'. This could be written as a formula, $S = (n - 2)\,180°$

This experiment is the example of the inductive method. When using this method, it is necessary to repeat the measurements many times to see if the same pattern is always obtained. Of course, it is impossible to work with all possible geometrical figures, and sometimes our measurements are not exact. This means that we cannot be sure that our conclusion will apply to every geometrical figure. Thus, the inductive method gives an answer that is only probably true.

However, we can arrive at a conclusion by using a different method. Let us divide the geometrical figures into triangles as shown in the following figure:



Here, we make the assumption that a geometrical figure can be divided into $(n - 2)$ triangles, a number of triangles that is 2 less than the number of sides. From our previous experiment with triangles, we assume that the measures of the angles of a triangle add up to 180°. It is also assumed that the angles of any geometrical figure will be made up of the sum of the angles of the triangles into which it can be divided. Then, we can conclude that the measure of the angles of any geometrical figure is $(n - 2)$ 180°.

Reasoning: Let us examine the reasoning just used. We started with several ideas that either assumed to be true or had previously established. Then, we use these ideas to reach a conclusion by the force of reasoning. No measurements were made. By starting with an assumption about the number of triangles in any geometrical figure, we found a specific conclusion about the angles of any geometrical figure. Such a method of logical reasoning is called deductive reasoning. By deductive reasoning we obtain a specific conclusion from other ideas or assumptions.

Of course, the truth of our conclusion depends upon the truth of the starting assumptions and ideas. Here is another, very simple example of deductive reasoning. Suppose, we assume every pupil in the fourth standard takes mathematics. If we know that Anshu is in the fourth standard, we can deduce that Anshu is taking mathematics. This specific conclusion is dependent upon the assumption with which we started our reasoning. Of course, we can be certain that our conclusion is true only if our original assumption that all fourth formers take mathematics is true.

Reasoning, Logic and Proof in Mathematics

We have examined two types of reasoning used in mathematics. We will see that both are useful, but both also have some drawbacks. Inductive reasoning is useful in making discoveries. If the examples considered are not representative or are misinterpreted then it can lead to false conclusions. Deduction will produce correct conclusions when you start with correct assumptions. These two methods are often used together in mathematics: induction to develop acceptable assumptions, deduction to derive true conclusions from the assumptions.

Man's first experience with mathematics was of an inductive nature. The ancient Egyptians and Babylonians developed many mathematical ideas through observation and experimentation and made use of this mathematics in their daily life. Then the Greeks became interested in philosophy and logic and placed great emphasis on reasoning. They accepted a few basic mathematical assumptions and used them to prove deductively most of the geometrical facts we know today. Hence of deductive proof became an important part of mathematics.

Since the time of ancient Greeks, deduction has been the most important type of reasoning used in mathematics.

However, mathematicians, like scientists, still discover new ideas from hunches, intuition, analogies, guesses and experiments. Then they work out rigorous proofs to check the truth of new ideas. This formal proof leaves nothing to the imagination. They use assumptions and definitions and previously proved statements to prove new statements. Usually they do not say, 'Such and such is true'. Instead they make statements like, 'If A is true, then B is true'. And they realise that the conclusion B, depends upon the starting assumption A, and might be true only in the world of mathematics with no apparent application or illustration in the physical world. For example, by using a logical proof, two Polish mathematicians, Stefan Banach and Alfred Tarski, have proved, from a mathematical stand point, that a solid sphere of the size of a pea could be divided into a finite number of pieces and then reassembled into a sphere of the size of the sun. No wonder mathematics is considered an unusual science.

Some Unsolved Mathematical Problems

Although we usually think that mathematics is an 'exact science' which solves all problems, but there are a number of mathematical problems, that are still mysteries to mathematicians.

1. **The Prime Number Mysteries:** No one has been able to write a formula or system that will test whether or not a given number is prime number. There must be some way of forming prime numbers, but no one has yet been able to find a systematic way to do it. Another mystery about prime numbers is asked by the question, 'Is there an infinite number of prime pairs?'

   A prime pair is a pair of prime numbers whose difference is 2; for example (3,5), (11,13), (41,43). These prime pairs seem to occur throughout our number system.

   No one has been able to find how many there are to discover a formula to locate them. But on the other hand, no one has been able to prove that there is a number beyond which there are no prime pairs.

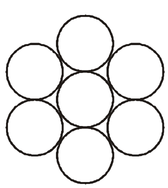
2. **Goldbach's Conjecture:** Is every even number the sum of two primes? C. Goldbach wrote a letter to Leonhard Euler, in which he made the conjecture that every even number except 2 was the sum of two primes. This was an interesting statement that was true for every even number he examined, but he could not prove that it was a true statement for all even numbers.

   For example $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$. No even number has been found that is not the sum of two primes. But these is no proof that every even number is the sum of two primes.

3. **The Odd Perfect Number Mystery:** The ancient Greeks considered some numbers to

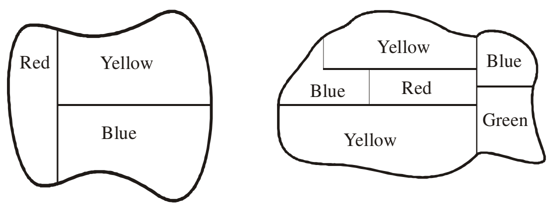
be perfect. Perfect numbers are numbers which are equal to the sum of their divisors. The number 6 is such a number because 6 = 1 + 2 + 3, 28 = 1 + 2 + 4 +7 + 14. Others have been found and all of them are even numbers. No one has ever found an odd perfect number. But no one has been able to prove that every perfect number must be even.

4. **Three Construction Problems:** Some of the first unsolved problems in mathematics were these three famous constructions proposed by the Greeks, to be solved using only a pair of compasses and a straight edge:
   (i) Can you draw a circle with the same area as a square?
   (ii) Can you drawa cube exactly twice the volume of a given cube?
   (iii) Can you divide an angle into exactly three equal angles?
5. **How to Pack Spheres:** A geometry problem that is still unsolved involves the packing of spheres such as ping-pong balls. How should spheres be packed in a box so that they use the least possible space? This is similar to a problem of drawing circles. How should circles be drawn or round objects like pennies be packed to cover the least surface? The arrangement for the circles on a surface has been found to be of the following pattern:



In packing spheres, this is also the best arrangement for the first layer. But nobody has solved the problem of how to arrange the second layer of the spheres.

6. **The Four–colour Map Problem:** There are also unsolved problems in the field of topology. One of them is the four-colour map problem. How many different colours are needed to make a map so that countries with a common border are coloured differently? The drawing below illustrates some possible maps.



This is a real mystery to map makers and to mathematicians. They have not been able to draw a map that needs more than four colours. But at the same time they have not been able to prove that four colours are enough for any possible map. However, technological proof is available for this problem.

Some Problems on Mathematical Thinking

1. Write the squares of the whole number from 1 to 10. Beginning with 2, write all numbers that can be obtained as sum of two or three or four of these squares.

   For example, 13 = 9 + 4 and 33 = 16 + 16 + 1 = 25 + 4 + 4. Can you find a number less than

100 that requires more than four squares as a sum?

Every number can be written as the sum of 4 squares or less

2. As $3^2 \times 3^2 \times 3^2$ cube painted red on all sides is cut into 27 cubes $1^2 \times 1^2 \times 1^2$. How many of these cut cubes are painted on 3 sides? 2 sides? 1 side? No sides? How did you solve the problem—by experiment or by using your imagination?

8 cubes have 3 sides painted.

12 cubes have 2 sides painted.

6 cubes have 1 side painted.

1 cube has no side painted.

3. Write all the prime numbers from 1 to 50. Find the differences between the primes. Can you find a formula that will give you any prime?

There is no formula that will give the prime numbers.

4. (a) What prime can you find less than 100?
   (b) How far apart are these prime pairs?
   (c) Are these prime pairs equally spaced?

(a) (3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73)

(b) 0, 4, 4, 10, 10, 16, 10

(c) No

5. What two prime numbers have each these even numbers for their sum?

a. 8 b. 26 c. 18 d. 48

a. 5 + 3 b. 19 + 7 c. 11 + 7 d. 37 + 11

6. Do these sets of number satisfy the equation $x^2 + y^2 = z^2$?
   a. $x = 3, y = 4, z = 5$
   b. $x = 7, y = 24, z = 25$
   c. $x = 8, y = 6, z = 10$

a. Yes b. Yes c. Yes

7. Do these sets of numbers satisfy the equation $x^3 + y^3 = z^3$?
   a. $x = 1, y = 2, z = 3$ b. $x = 2, y = 3, z = 4$

a. No b. No

8. A girl once said: 'The man I marry will be tall, not dark, quite heavy, will wear glasses, will drive fast cars, and will be an Indian'.

   Amar is fair, tall Indian and wear glasses but does not drive fast cars.

   Apurva is not short, wears glasses, drive fast cars, is not fair, is not heavy and is an Indian.

   Anshuman drives fast cars, is dark and not too heavy, is not short and is certainly not fair. He wears glasses.

   Which one of the three men will the girl marry?

The girl will marry Amar because, he has more of the required qualities than any other man. She will marry none of them, if she insists that the man have all of the qualities she is looking for.

9. What is the limit of the unending decimal 1.99999...?

   Answer : 2

10. How many fractions are there in between 0 and 1 ?

    Answer : Infinite number

11. What is the limit of $1\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}-......?$

   Answer : 0

12. Find examples of some number or measurements that are unending?
13. Which one of the following would be the best sample to use deciding what car is most popular in your town?
    a. A sample of the owners of Jaguars.
    b. A sample of the women drivers in your town.
    c. A sample of the people who do not have own a car.
    d. A sample of the people who bought a car during the last year.
    e. A sample of everyone in your town. (Answer c)
14. Collect data from your newspaper or school or friends to find the answers to questions like these:
    a. How many hours do your classmate spend watching TV each week?
    b. How the average score in rugby games changed for your school in the past 5 years?
    c. What is the average height of your friends?
    d. How many of your school go on to university each year?
15. Here are some data for an object falling to the earth:

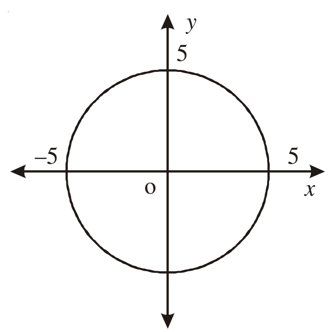
   What formula shows the relationship of S and t?

Time in seconds (t)	Distance fallen (s)	t^2
1	16	1
2	64	4
3	144	9
4	256	16

$S = 16t^2$

16. On the graph paper, draw the geometrical figure for the equation $x^2 + y^2 = 25$ from the following data:

x	y	x^2	y^2
0	5	0	25
1	4.9	1	24 (approx.)
2	4.6	4	21 (approx.)
3	4	9	16
4	3	16	9
5	0	25	0



About Different Branches of Mathematics

Each branch or field of mathematics has been developed and organised logically. For this reason, each field of mathematics is called a *logical structure or system*. To build a mathematical system, the mathematician begins with a set of undefined words and a set of unproved assumptions. He must have words to express his ideas and assumptions from which he can deduce his ideas. He then develops rules that describes the things that can be done with the mathematical system. Next, he invents symbols to simplify the use of the rules. These words, symbols or rules do not need to have any reference to objects of the world around us. Mathematics often includes things that have never been seen or experienced. Some persons think that all mathematics have to be verified by concrete objects, measurements or experimentation. This is not so, for mathematics belongs to the realm of ideas, imagination and fantasy. But, mathematics is unique in that it is also one of the most practical studies known to man.

Arithmetic: A Sample Mathematical Structure

Let us see how elementary arithmetic illustrates a deductive system. Arithmetic begins with the number 1 as an undefined term. With this unit, all other numbers can be defined. The operations of addition and multiplication are also undefined terms. We describe these operations with addition and multiplication tables, but we do not give a definition of them. However, we define the operations of subtraction and division in terms of addition and multiplication, respectively.

Next, we state some assumptions about numbers that seem to agree with our experiences. For example, we assume the associative law to be true, that is,

$$2 + (3 + 4) = (2 + 3) + 4$$

We also assume that the sum of two numbers is always another number.

Then we define our numbers in terms of the unit 1 and addition:

$2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$, and so on.

These undefined terms, undefined operations, assumptions and definitions are then used to prove new relationships called *theorems*. Let us see, how we can use this deductive system to prove that $2 + 2 = 4$.

A Sample Logical Proof

Statements	Reasons
1. $2 + 2 = 2 + (1 + 1)$	1. 2 is defined as equal to 1 + 1
2. $2 + (1 + 1) = (2 + 1) + 1$	2. Associative law
3. $(2 + 1) + 1 = 3 + 1$	3. 3 is defined as equal to 2 + 1
4. $3 + 1 = 4$	4. By the definition of 4

In a similar manner, we can prove most relationships in arithmetic that we usually take for granted. This method of deduction can be applied to other mathematical systems and concepts. In each case, the mathematical structure developed is based on undefined terms, assumptions, definitions and the proofs of theorems.

Geometry

The mathematical study of space, shape and measurement is know as geometry. Geometry tells us how to draw different shapes or figures and tells us many facts about the relationships among these figures.

The study of geometry has been basic for artists, engineers and architects. Drawing accurate plans for a building, determining the effect of a strong wind on an aeroplane, painting a picture that has a balanced and pleasing design, all are related to the ideas about points, lines, angles and shapes that are studied in geometry.

For centuries, man was curious about geometrical mysteries such as the relationship between the diameter and circumference of a circle, whether parallel lines ever meet, whether space is curved and without a boundary. Even such a simple idea as a line has been interesting. You see, no one has ever seen a line. In geometry, a line has no width — only length. And no one knows if it is possible for a line to be 'straight'.

New Geometries

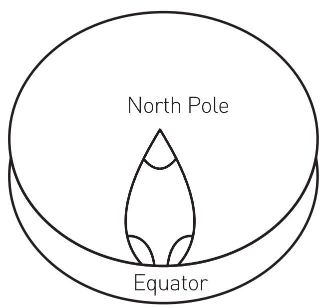
The geometry with which we are most familiar is called *Euclidean geometry*. This name is a tribute to the Greek scholar Euclid (323-285 B.C.), who collected, systematised and recorded the geometrical knowledge of his day. However, Euclidean geometry is not the only possible kind of geometry. By starting with some assumptions about lines that differ slightly from the assumptions of Euclid, mathematicians in the last century developed some strange geometries, all are called non-Euclidean geometries.

One non-Euclidean geometry says that the shortest distance between two points is not a straight line, but, rather, a curved line. Does this sound strange? It does if we are talking about flat surfaces. But this idea makes sense, if we consider a sphere or a ball instead of a flat surface. Indeed, navigator ships and aircraft know that the shortest distance between two points on the surface of the earth (approximately a sphere) is a curve which we call a great circle.

With this idea of a great circles, we can go a step further and reach another astonishing conclusion of non-Euclidean geometry, that the measures of the angles of a triangle drawn on a sphere may add up to more than 180 degrees.

Suppose that the sphere is the earth. The base of the triangle is the equator. The two lines intersecting the equator are meridians, meeting at the North Pole and forming the triangle. The sum of the measures of the angles of this triangle is equal to the two 90° angles (spherical angles are measured by the degrees in their arcs) plus the angle formed at the North Pole.

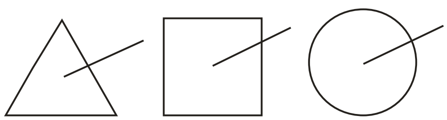
Another kind of new geometry is one called *topology*. Unlike the geometry we usually study, topology is not interested in size or shape. Topology deals with lines, points and figures but these elements are allowed to change in size and



shape in topology. Sometimes topology is called the *rubber-sheet geometry* because topological figures can be stretched and twisted, as they might be on a rubber-sheet, and still remain the same, topologically speaking.

Topology is concerned with properties of position that are not related to size or shape. In the world of topology, a straight line like AB below is the same as the curved line CD because each of them represents a path between two points.

Topology also says that a triangle, a square and a circle are the same. They all have one inside and one outside and to go from the inside to the outside you must cross one and only one line.



These new ideas of topology have been the means of solving many problems in mathematics.

Exercise

An explorer walks 1 km south, turns and walks 1 km east, turns again and walks 1 km due north. He finds himself back where he started. At what places on the earth's surface is this possible? (There is more than one place where this is possible).

[Answer: At the north pole or about 1 km for the south pole.]

Algebra, the Language of Mathematics

One of the things about mathematics that makes it such a powerful science is its use of symbols such as + or –. These symbols make mathematics a shorthand language. It is much simpler to write 5+3 =8 than to write 'the sum of five and three is eight'. Such symbols make it easy to think about mathematical ideas, and make it possible to show that ideas or relationships which are true for specific numbers may also be true for all numbers. For example, we know that 5 + 3 = 8 and 8 – 3 = 5. If we use *a*, *b* and *c* to represent any numbers, then we can say that $a + b = c$ then $c - b = a$. This use of letters as placeholders for numbers is one of the big ideas of algebra. You have probably used letters as placeholders in a similar way in formulas like $c = pd$, the formula for the circumference of a circle.

Formulas like the one above, $c = pd$, are called mathematical sentences. In algebra you learn how to find the value for placeholders like *c* and *d* which make a sentence true. In the sentence, $x + 3 = 8$, what number can replace *x* so that this is a true sentence?

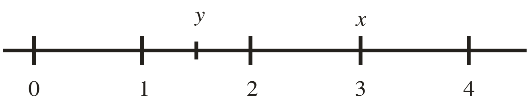
When a scientist performs an experiment to discover new facts, he often arranges his measurements in the form of a table. Then he tries to write a mathematical sentence like a formula that shows how the numbers in his table are related.

Let us look at the following set of data to see whether it can be expressed by a formula.

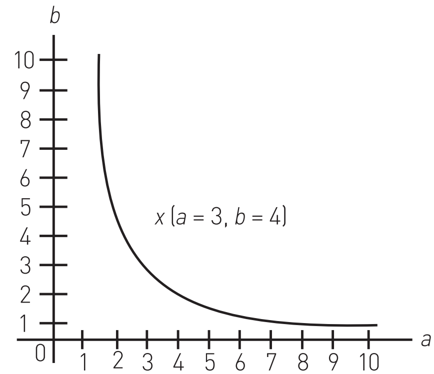
Measurement a	*Measurement* b	a + b	a × b
1	12	13	12
2	6	8	12
3	4	7	12
4	3	7	12
6	2	8	12
8	$1\frac{1}{2}$	$9\frac{1}{2}$	12
10	$1\frac{1}{5}$	$11\frac{1}{5}$	12
12	1	13	12

If we compare the measurements by addition ($a + b$), we do not find a pattern for the sums. But if we look at the products of the measurements ($a \times b$), we find that they are always 12. This tells us that formula for this pattern is $a \times b = 12$. In this way, algebraic ideas are used to discover and state relationships.

A famous French mathematician by the name of Rene' Descartes (1596 – 1650) showed how algebra and geometry are related. He drew a line to show that any number could be represented by a point on the line and that any point on the line could represent a number. For example, in the following figure, point x represents 3 and point y represents 1.43786



With two number lines drawn perpendicular to each other, we can use a point to represent a pair of numbers. For example, point x in the following figure represents the number pair $a = 3$, $b = 4$. If we locate points for the number pairs for a and b given in the table and join the points with a curved



line, we get a part of a geometrical figure called the hyperbola as shown in the following figure:

Thus, the mathematical sentence, $a \times b = 12$, can be represented by a geometrical figure.

Some Magic with Algebra

Algebra is best known for its use in solving problems. This is illustrated by this number trick: Think of a number between 0 and 10. Multiply it by 5.

Instructions	Your friend's arithmetic	Your solution with algebra
Think of a number between 0 and 10.	Think of 6	Use x as a placeholder for 0 this unknown number.
Multiply it by 5	$5 \times 6 = 30$	$5x$
Add 7 to your answer	$30 + 7 = 37$	$5x + 7$
Multiply this answer 2	$37 \times 2 = 74$	$10x + 14$
Add any other number between 0 and 10	Pick 8; $74 + 8 = 82$	Use y as a placeholder $10x + 14 + y$
Subtract 3 from this result	$82 - 3 = 79$	$10x + (14 - 3) + y = 10x + 11 + y$
If you tell me your answer, I will tell you what numbers you choose	79	6 and 8

Add 7 to your answer.

Multiply this number by 2.

Add any other number between 0 and 10.

Subtract 3 from this result.

If you tell me your answer, I will tell you what numbers you choose.

Here, how this trick works with a friend?

How can you get the unknown numbers 6 and 8? You set your algebraic expression equal to the number by your friend:

$10x + 11 + y = 79$

Then subtract 11 from your expression and your friend's answer to get

$10x + y = 68$

Now, you know that $10x$ is 10 times the number he choose first, or the tens digit 6, and the symbol y represents the number he added to the tens digit or 8. Try this trick with your friends to demonstrate the magic you can do with algebra.

Probability: The Science of Chance

One of the most intriguing fields of mathematics is the study of probability or chance. Probability tells us how often we may expect some event to occur. It is the field of study which is giving the scientists and the social scientists the tools to deal with our uncertain world. Calculating the probability that something will happen is like looking into the future.

Probability is used in business, in government, and in science to make predictions. Using probability, an insurance company predicts how many houses are likely to burn every year, civil servants predict how much income tax will be collected next year, and a scientist predicts the performance of a space ship before it takes off. Since we all run the risk of accidents, difficulties or success every day in our life. It is important that we know something about the mathematics of probability.

A simple illustration of probability is the tossing of a coin. If you toss a coin, it is just as likely to turn up heads as tails. When we toss several coins, we expect about half of them to turn up heads and the other half tails. So we say your chance of getting a head when you toss one coin is $\frac{1}{2}$. In more technical language, we say that the probability of an event happening is equal to the ratio of favourable ways to the total number of ways it could happen.

To see how this works with several coins, we need to study all the possibilities. If we toss two coins, the coins may fall four ways, as shown below:

H	T	H	T
H	H	T	T

Since one of the four ways gives us 2 heads, we say the chance of tossing 2 heads with 2 coins is $\frac{1}{4}$.

If we toss 3 coins, the coins may fall eight ways, like this:

H	H	H	H	T	T	T	T
H	H	T	T	H	H	T	T
H	T	H	T	H	T	H	T

What are the chances of getting 3 heads? What are the chances of getting 2 heads and 1 tail?

Statistics: Making Sense out of Data

Number facts called data or numerical statistics are found in every daily newspaper. In industry, on the farm, in state and national legislatures, at home, in school, and at church, statistics plays an important role. Information about the weather, sports, jobs, wages, prices and population changes is daily news. These data are the basis for important decisions. If these decisions are to be the right decisions, the data must be read and interpreted intelligently.

An important application of statistics is the use of sampling. Sampling is used in public opinion polls, television programmes popularity ratings and quality control in industry. In sampling, we choose a small number of items which we think are typical of the whole and examine the sample. We can then say, if our sample has been properly chosen, that what is true of the sample is also true of the whole.

Here is a simple example of how sampling works. Suppose you wish to find out something about the size of the pebbles on a beach. Obviously, you cannot measure all pebbles. Instead, you might select a sample of, say, 500 pebbles from all parts of the beach. The statistician has determined that 99 times out of 100 only 1 per cent of all the pebbles on this beach will be larger than the largest pebble in the sample, so you can get good picture of the size of all the pebbles from those in your sample.

This same type of result is expected whether the sample is of people, fish, flowers or numbers drawn out of a hat. An application of this method in industry might be to test samples of tyres to determine the quality of all the tyres produced. By testing a proper sample of 65 tyres, the manufacturer can conclude that 90 per cent of all the tyres in the sample will almost certainly have a longer wearing life than the second tyre to wear out during the test. By the way, statistician calls 99 chances out of 100 "almost certain".

Statistical methods, such as the presentation of data in tables and graphs, help us to read and interpret data. Statistics also tells us how to calculate measures such as averages or deviations that tell us the trend of the data. Relationships are determined by finding a measure of association called the correlation coefficient. Then probability is used to state the meaning of the data or to predict a result in the future. For example, a statistician may look at your mathematics test scores, tell you how to compare with an average student, predict your success in college, and tell you the probability that his prediction is right.

Infinity, Limits, Changing Quantities and Calculus

How far is it to the end of space? How long ago did time begin? How many numbers are there in our number system? How many points are there on a line? The answer to all these questions is "an infinite number". By this, we mean that the number of items in the answer cannot be counted.

Infinity is an idea to understand and impossible to visualise. But mathematicians have invented ways of dealing with relationships and situations that involve an infinite number of terms. Suppose we have the series of fractions, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ and so on, each new fraction being formed by taking half of the preceding fraction. What is the sum of all the fractions of this form that you could write?

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+....$$

There is no end to the number of fractions like this that you could write. We say there would be an *infinite* number of fractions. If you find the sum of large number of these fractions, you will find that the sum is very close to 1. Mathematicians say that the limit of the sum of these fractions is 1 because you can write fractions like this until the sum is as close to 1 as you desire. However, the sum will never be as much as 1. Whenever we work with infinite amounts of things such as time, points, or numbers, we use special kinds of mathematics. One field of mathematics that deals with infinity and limits is *calculus*.

Calculus is the mathematics that enables us to study the relationship between changing quantities. Here is a typical question that has been answered by calculus:

If a pebble is dropped from a cliff, at what speed would the pebble be travelling in 6 seconds?

In this problem, the distance changes as the time changes and the speed also will change as the time changes. The methods of calculus have been used to solve this problem by finding a *limiting value* for the speed as the time divided into smaller and smaller units. These methods would finally tell us that the speed at any instant would be equal to 9.8 times the number of seconds travelled. In 6 seconds, the pebble would be travelling at a speed of 58.8 meter per second.

Sets: A Useful Mathematical Idea

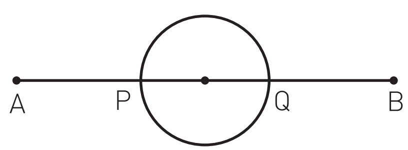
One of the newest ideas in mathematics developed in the past century is set theory, invented by the German mathematician George Cantor (1845-1918). It has been a means of finding new facts and of proving old facts.

The ideas of a set is very simple. A set is a collection of objects, numbers, persons or ideas. You are already familiar with sets such as a set of books, a set of dishes, or a set of tools. You are even a member of a set of people such as your mathematics class, your scout troop or your family.

Set ideas are used in many fields of mathematics. In arithmetic, we talk about set of numbers. For example, the set of prime numbers less than 10 is {2, 3, 5, 7}.

In algebra, we talk about the solution sets for sentences. For example, the solution set for $x - 7 = 5$ is the number 12.

In geometry, we talk about the set of points that meet certain conditions. The set of points on line AB and also on the boundary of the circle are the points P and Q.



In statistics, we talk about sets of data. For example, the set of scores on the weekly mathematics quiz was {7, 10, 5, 6, 9, 7, 8, 4, 6, 8, 5, 7}.

In probability, we talk about the set of all possible events. For example, the set of ways three letters, A,B, C can be arranged is {ABC, ACB, BAC, BCA, CAB, CBA}.

In measurement, we talk about a set of units of measure. A set of metric units of length is {kilometer, hectometer, decameter, meter, decimeter, centimeter, millimeter}.

Set theory has been the basis for new philosophy about mathematics. It has freed mathematics from working with individual numbers, permitting it to consider sets of numbers. Set ideas have also made it possible to solve many problems that involve relationships rather than numbers. Sets have also been proved useful in setting up problems for electronic computers.

Developments in New Mathematics

Although mathematics is one of the oldest sciences, it is growing like a boy at school. New ideas, new vocabulary, new methods, new problems are constantly being created. Mathematics has never been changing as rapidly as it is today.

One of the most exciting new fields of mathematics is *game theory*. The mathematician John von Neumann contributed a great deal to this theory. Game theory is a mathematical way of describing and analysing competition among groups of people such as clubs, athletes or draughts players. Game theory gives a mathematical analysis of what decisions or moves should be made to 'win' in a competitive situation. Thus, game theory can be used to make decisions in business, in foreign affairs, and in the detection of a murderer.

New electronic computers and new ways of solving problems with computers have been designed by mathematicians. These computers have been used to design automatic control of factories. These computers have also been used to extend mathematical knowledge; for example, a computer has made a list of all prime numbers below 46,000,000. However, a computer cannot create new mathematical ideas, and cannot solve the simplest problem unless a mathematician first 'tells' it how to solve the problem.

A new algebra that applies to motions, numbers, or space is called *group theory*. We have already mentioned a new geometry, topology, that is concerned with relationships of points and lines but is not concerned with shape or size. A new arithmetic has been created to compute with new numbers called *quaternions*. *Sets* are being used in a new way to solve logical problems with electric circuits.

How does a mathematician create new mathematical ideas? New mathematics often comes from plain curiosity about a problem or an idea. By intuition, estimation, guess, experimentation, recollection, visualisation, the mathematician searches for a clue to the mystery. *Experiments in mathematics often require no tools, no equipments and usually no materials* *other than paper and pencil*. From the results, the mathematician states a conclusion. Then he uses deduction to prove that his conclusion is correct.

New Uses for Old Mathematics

Although we have noted many practical uses for mathematics, mathematicians are not always concerned with applications when they produce new mathematics. When Gottfried Leibnitz (1646-1716), the great German mathematician, created the binary numeration system, in which we can write any number using only the figures 0 and 1, is the one that is used in the operation of electronic computers. When the existence of a number such as $\sqrt{-5}$ was suggested, it was called an *imaginary number* because it seemed such a ridiculous idea. After all, it was thought, there is no number which, when multiplied by itself, produces – 5. But now imaginary numbers are needed to solve the problems about electricity. When the German mathematician Bernhard Riemann (1826 –1866) suggested that the shortest distance between two points is a curved line, it was considered a foolish statement. Now, nuclear physicists are using this idea and other ideas of non-Euclidean geometry in their research.

Throughout the study of pattern, mathematicians have discovered ideas that later have been found very useful. For example, the Greeks studied the ellipse over a thousand years before the German astronomer and mathematician Johannes Kepler (1571-1630) used their ideas to predict the motion of the planets. Now, the formulas for predicting the motion of planets are applied to artificial satellites and will be useful in acquiring knowledge about space travel.

The new mathematical ideas being created today may not have application for years or even centuries. But most mathematicians are confident that if they create good mathematics it will someday be found useful. The Polish-American mathematician, Samuel Eilenberg, illustrates this when he compares his work with that of a creative tailor. He says, "Sometimes I make coats with five sleeves, other times with seven sleeves. When it pleases me, I make a coat with two sleeves. And if it fits someone, I am happy enough to have him wear it".

The Power of Mathematics

In this brief journey into mathematics, we have tried to throw a spotlight on some of the interesting scenery that lies along the road. When you travel a road more slowly in the work that you will do in mathematics classes, you will have an opportunity to look more carefully at the trees, the plains, the structures lining that road. But even in this quick trip, we hope that you have been able to see that these are some of the remarkable qualities that mathematics possesses:

Mathematics provides an ideal model for logical reasoning: The deductive logic of mathematical proofs is considered the best method known for proving that a new idea is true. When the inductive method is used to establish facts, the conclusion is usually stated in terms of the mathematical theory of probability.

Mathematics has clarity and conciseness of expression: For example, the expression $\{x \mid x + 3 = 5\}$ is a short way of saying, "find all values of x which make the sentence $x + 3 = 5$ a true statement".

Mathematics has novelty and a variety of fields: For instance, 2 + 3 = 5 is a truth that always has been true and always will be true as long as 2, 3 and 5 are counting numbers. Mathematics may be concerned with such different things as maps, motion, games, music, arts, probability, infinity, philosophy and numbers. Who would ever expect that a mathematician would be interested in paper folding? But the folding of paper flexagons has intrigued many mathematicians.

Mathematics has abstractness which adds to its power: Geometry discusses points and lines, but no one has even seen a point or a line. No one can make an object that has more than three dimensions, but mathematicians use expressions for the fourth dimension. No one can write all the prime numbers, but mathematicians can prove that there are an infinite number of prime numbers.

Mathematics has the power to predict events: In 1905, Einstein was able to write the formula that predicted the amount of energy that would be obtained by an atomic explosion. In astronomy, it is possible to predict an ellipse of the Sun using the formulas that give the motion and position of the heavenly bodies.

Mathematics can measure amounts indirectly: Mathematicians have measured the distance to the Sun and the temperature at the middle of the Sun without getting closer than about 92000000 miles (14,72,00,000 km). Even in 230 B.C., Eratosthenes measured indirectly the distance around the Earth before it was even known that the earth was round.

Mathematics has unlimited opportunity for creativeness: Just as no one has composed the most elegant poem or painted the most beautiful picture, so no one has invented the ultimate mathematical structure. Even field of mathematics, from arithmetic to topology, gives ample opportunity for the creation of new ideas.

Mathematics has more permanence than any other field of knowledge: It is the only science in which the measure theories of 20 centuries ago are still true and useful. The value of π has always been 3.14159... and always will be (even a state legislature once tried to change it to 3!).

The musical scale that the ancient Greek philosopher and mathematician Pythagoras established, in which vibrations producing tones have the ratios $\frac{1}{2}, \frac{1}{3}$ and $\frac{3}{4}$, continuous to be a basic scale in music.

Mathematical curves and surfaces have a balance and symmetry that are as pleasing as a masterpiece in art: From the simple circle to the complex hyperbolic paraboloid, from the golden section to the pendulum patterns, mathematical curves are basic in art and architecture, in advertising and in the arts.

Mathematics is found in the designs and laws of nature: From the spiral of a snail's shell to the symmetry of a snowflake, from the bee's hexagonal cell to the elliptical orbits of the planets, mathematical curves and geometrical designs occur in the world of nature. Similarly, the distance of fall of a raindrop $\left(s = \frac{1}{2}gt^2\right)$ and energy in an atom ($E = mc^2$) can be expressed by mathematical formulas.

Conclusion

Thus, in the preceeding paragraphs, we have discussed:

- Role of inductive and deductive reasoning and explaining basic concepts/formulas involving activity.
- Role of patterns in mathematics leading to conjectures and generalisations.
- Use of mathematics in handling problems in interdisciplinary areas.
- Use of examples and exercises to sharpen mathematical reasoning.
- Developmental process of some fundamental concepts/branches in historical perspective.